Call $E \subseteq \mathbb{R}$ measurable ($E \in \mathfrak{M}$) if the following condition of Carotheodory is satisfied

(*) $m^*(A) = m^*(A \cap E) + m^*(A \cap \widetilde{E}) \quad \forall A \subseteq \mathbb{R}$

By the subadditivity of $m^*$, this is the case iff

(**) $m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap \widetilde{E}), \quad \forall A \subseteq \mathbb{R}$ with $m^*(A) \neq +\infty$.

e.g. $E \in \mathfrak{M}$ if $m^*(E) = 0$ (so $m^*(A \cap E) = 0$ and (**) holds by the ↑-property of $m^*$). Note also that $E \in \mathfrak{M} \Leftrightarrow \widetilde{E} \in \mathfrak{M}$.

Will show in this section that $\mathfrak{M} \supseteq \mathcal{B}$ (Borel sets) and $m := m^*|_{\mathfrak{M}}$ is a measure:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n), \quad \forall E_n \in \mathfrak{M} \ (n \in \mathbb{N}) \ \& \text{ disjoint}$$

(already know $m(\emptyset) = 0$). We do this in steps (in a series of lemmas).

L1. $\mathfrak{M}$ is an algebra (of subsets of $\mathbb{R}$).

Pf. Need only show that $E \in \mathfrak{M}$ if $E = E_1 \cup E_2$ & $E_1, E_2 \in \mathfrak{M}$.

To do this, let $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$. Then (**) holds as

$$\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \cap \widetilde{E_1}) \quad (\because E_1 \in \mathfrak{M}) \\
&= m^*(A \cap E_1) + \left[ m^*((A \cap \widetilde{E_1}) \cap E_2) + m^*((A \cap \widetilde{E_1}) \cap \widetilde{E_2}) \right. \quad (\because E_2 \in \mathfrak{M}) \\
&= m^*(A \cap E_1) + m^*((A \cap \widetilde{E_1}) \cap E_2) + m^*(A \cap \widetilde{(E_1 \cup E_2)}) \\
&\geqslant m^*((A \cap E_1) \cup (A \cap \widetilde{E_1}) \cap E_2) + m^*(A \cap \widetilde{(E_1 \cup E_2)}) \\
&= m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \widetilde{(E_1 \cup E_2)})
\end{aligned}$$

because

$$E_1 \cup E_2 = E_1 \cup (\tilde{E}_1 \cap E_2) \quad \left(\overset{\text{why}}{\text{check!}}\right)$$

Ex. Can you directly check (similarly) that

$$E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \cap E_2 \in \mathcal{M}$$

L2. Let $E_1, E_2 \cdots E_n \in \mathcal{M}$, disjoint. Then

$$(\#) \quad m^*(A \cap (E_1 \cup \cdots \cup E_n)) = \sum_{i=1}^{n} m^*(A \cap E_i), \ \forall A \subseteq \mathbb{R}.$$

( If $E_1 = E \in \mathcal{M}$ and $E_2 = \tilde{E}$ then (#) is simply (*)).

pf. By MI with the crucial step:

$$\text{LHS of } (\#) = m^*((A \cap (E_1 \cup \cdots \cup E_n)) \cap E_n) + m^*((A \cap (E_1 \cup \cdots \cup E_n)) \cap \tilde{E}_n) \quad (\because E_n \in \mathcal{M})$$

$$= m^*(E_n) + m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right) \quad (\because \text{pairwise disjoint})$$

$$= m^*(E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) \quad (\text{induction assumption})$$

L3. Let $\{E_1, E_2, \cdots\} \subseteq \mathcal{M}$, pairewise disjoint. Then

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i) \quad \forall A \subseteq \mathbb{R}$$

pf. $\forall n \in \mathbb{N}$, one has by L2 that

$$\text{LHS} \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i), \quad \text{valid } \forall n \in \mathbb{N}$$

so LHS $\geq$ RHS. The converse ineq. holds by countable subadd.

L4. Let $\{E_i : i \in \mathbb{N}\} \subseteq \mathcal{M}$, pairwise disjoint

Then $E := \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

Proof. Let $A \subseteq \mathbb{R}$, with $m^*(A) < +\infty$. Then, $\forall n \in \mathbb{N}$, one has by L1 that

$$m^*(A) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*\left(A \cap \widetilde{\left(\bigcup_{i=1}^{n} E_i\right)}\right)$$

$$\geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*\left(A \cap \tilde{E}\right) \quad (\text{as } m^* \uparrow)$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E}) \quad (\text{by L2})$$

so

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E}) \quad (\text{why?})$$

$$\geq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad \left(\begin{array}{c}\text{countable}\\ \text{subadd.}\end{array}\right)$$

Thus (**) holds and $E \in \mathcal{M}$.

L5. Let $a \in \mathbb{R}$. Then

$$E := (a, +\infty) \in \mathcal{M}.$$

Pf. Let $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$. To show (#) it suffices to show that, $\forall\, \varepsilon > 0$,

$$m^*(A) + \varepsilon > m^*(A \cap \overbrace{E}^{(a,+\infty)}) + m^*(A \cap (-\infty, a))$$

(noting $m^*(A \cap (-\infty, a)) = m^*(A \cap \tilde{E})$). By def of $m^*(A)$, $\exists$ COIC $\{I_n : n \in \mathbb{N}\}$ of $A$ s.t.

$$m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \left(\ell(I_n') + \ell(I_n'')\right)$$

where $I_n' = I_n \cap (a, +\infty)$ & $I_n'' = I_n \cap (-\infty, a)$. Note that $\{I_n' : n \in \mathbb{N}\}$ & $\{I_n'' : n \in \mathbb{N}\}$ are COIC of $A \cap (a, +\infty)$ & $A \cap (-\infty, a)$ resp., so

$$\varepsilon + m^*(A) \geq m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a)),$$

as was required to show. [or use $\ell(I_n) = \ell(I_n \cap (a, \infty)) + \ell(I_n \cap (-\infty, a])$
$= m^*(I_n \cap E) + m^*(I_n \cap \tilde{E})$
so $\sum \ell(I_n) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$ ]

Th1. $\mathcal{M}$ is an $\sigma$-alg s.t.

$$E \in \mathcal{M} \quad \forall\, E \in \mathcal{B}$$
$$\quad \& \quad \forall\, m^*(E) = 0$$

and $m := m^*|_{\mathcal{M}}$ is a translation-inv. measure on $\mathcal{M}$ s.t. $m(I) = \ell(I)$ $\forall\, I \in \mathcal{M}$.

Pf. You can now do by applying the above lemmas together with following $\underline{\underline{\text{two}}}$ ex.

Ex 1. Let $\mathcal{A}$ be an algebra & stable w.r.t. countable disjoint unions. Then $\mathcal{A}$ is an $\sigma$-alg.

Sol. Let $E = \bigcup_{n=1}^{\infty} E_n$ with each $E_n \in \mathcal{A}$. Then

$E \overset{\text{why}}{=} \bigcup_{n=1}^{\infty}{}^{\circ} F_n$ where each $F_n = E_n \setminus \left( \bigcup_{i<n} E_i \right)$ ($\in \mathcal{A}$ as $\mathcal{A}$ is an alg)..

EX 2. Let $\mathcal{A}$ be an $\sigma$-alg (of subsets of $\mathbb{R}$) s.t.

$$(a, +\infty) \in \mathcal{A} \quad \forall a \in \mathbb{R}.$$

Then $\mathcal{B} \subseteq \mathcal{A}$.

Hint: $$[a, +\infty) = \bigcap_{\varepsilon > 0} (a-\varepsilon, +\infty) = \bigcap_{n=1}^{\infty} (a - \tfrac{1}{n}, +\infty) \in \mathcal{A}$$

$$(-\infty, a), \ (-\infty, a] \in \mathcal{A} \quad \forall a \in \mathbb{R}$$

$$(a, b) = \mathbb{R} \setminus \Big( (-\infty, a] \cup [b, +\infty) \Big)$$

Exercise 3. Let $(X, \mathcal{X}, \mu)$ be a "measure space":

$X$ is a set, $\mathcal{X}$ is an $\sigma$-alg of subsets of $X$, $\mu: \mathcal{X} \to [0, +\infty]$ a measure.

Then

(i) (Subtraction Lemma) If $A, B \in \mathcal{X}$ with $A \subseteq B$ then

$$\mu(B \setminus A) = \mu(B) - \mu(A) \text{ provided that } \mu(A) < +\infty.$$

(ii) (Monotone Convergence ($\uparrow$) Lemma) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{X}$ with $A_n \subseteq A_{n+1}$ $\forall n$. Then $\mu(A_n) \uparrow \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right)$:

$$\lim_n \mu(A_n) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right).$$

[Hint: Can assume $\mu(A_n) < +\infty$ $\forall n$].

(iii) ($\downarrow$-lemma) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{X}$ with $A_n \supseteq A_{n+1}$ $\forall n$. Then $\mu(A_n) \downarrow \mu(\bigcap_{n\in\mathbb{N}} A_n)$ provided that $\mu(A_1) < +\infty$ (or $\mu(A_N) < +\infty$ for some $N$).

(The details are to be provided at the end of this file)

Th2. (Littlewood's 1st Principle). Let $E \subseteq \mathbb{R}$. Then the following statements are equivalent (☺):

(i) $E \in \mathcal{M}$.

(ii) $E$ is outer regular: $\forall \varepsilon > 0$ $\exists$ open $G \supseteq E$ s.t

$$m^*(G \setminus E) < \varepsilon.$$

(iii) $\exists$ $G_\delta$-set $H \left(:= \bigcap_{n\in\mathbb{N}} G_n, \text{ each } G_n \in \mathcal{T}\right) \supseteq E$ s.t

$$m^*(H \setminus E) = 0.$$

(iv) $E$ is inner regular: $\forall \varepsilon > 0$ $\exists$ closed $F \subseteq E$ s.t.

$$m^*(E \setminus F) < \varepsilon.$$

(v) $\exists$ $F_\sigma$-set $K \left(:= \bigcup_{n\in\mathbb{N}} F_n, \text{ each } F_n \text{ closed}\right) \subseteq E$ s.t

$$m^*(E \setminus K) = 0.$$

Moreover, under the additional assumption that $\underline{m^*(E) < +\infty}$, each of the above also equivalent to

(vi) $\forall \varepsilon > 0$ $\exists$ $U := \bigcup_{n=1}^{N} I_n$ (disjoint open intervals) s.t.

$$m^*(E \triangle U) < \varepsilon. \left(\text{so } m^*(U \setminus E) \overset{<\varepsilon}{\&} \mu^*(E \setminus U) < \varepsilon\right)$$

Remark. The implication (vi) $\Rightarrow$ (ii) ($\&$ all (i)–(v)) holds regardless $m^*(E) < +\infty$ or not: By Th3 of the last subsection $\exists$ open $G_0 \supseteq E \setminus U$ s.t. $m^*(G_0) < \varepsilon$. Then $U \cup G_0$ is an open set containing $E$

and $(U \cup G_0) \setminus E \subseteq (U \setminus E) \cup (G_0 \quad )$

of outer measure $< \varepsilon + \varepsilon = \varepsilon$

so $E$ is outer-regular.

Proof (i) $\Rightarrow$ (ii). Let $\varepsilon > 0$. We do in two steps: special case when $m(E) < +\infty$ first. Then, by Th 3 of the preceding subsection, $\exists$ open $G \supseteq E$ s.t. $m^*(G) \left( = m(G) \right) < m(E) + \varepsilon$. Noting

$$G = E \cup_0 (G \setminus E)$$

and so

$$m(G) = m(E) + m(G \setminus E)$$

and $\quad m(G \setminus E) = m(G) - m(E) < \varepsilon$.

Next, consider the general case: $m(E) \leq +\infty$.

Let $E_n := E \cap (-n, n) \quad \forall n \in \mathbb{N}$. By the preceding para. applying to $E_n$ (with $m(E_n) < +\infty$), $\exists$ open $G_n \supseteq E_n$ s.t. $m(G_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Let

$$G := \bigcup_{n=1}^{\infty} G_n \quad (\in \mathcal{T})$$

Then $G$ is an open set containing $E$

and $G \setminus E \subseteq \bigcup_{n \in \mathbb{N}} (G_n \setminus E_n)$ of mea $< \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)
$\Updownarrow$ $\nearrow$
(iv) $\Rightarrow$ (v)

Easy Exercises!

$\therefore$ (i) – (v) are mutually equivalent

It remains to show that

(i) $m(E) < +\infty \Rightarrow$ (vi). (so (ii) holds) Let $\varepsilon > 0$. Then, as before, $\exists$ open $G \supseteq E$ s.t. $m(G) < m(E) + \varepsilon$. By the structure Th for open sets $G := \bigcup_{n \in \mathbb{N}} I_n$ with disjoint open intervals $I_n$ $\forall n$. Then $+\infty > m(G) = \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \ell(I_n)$

so $\exists N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} \ell(I_n) < \varepsilon$. Let $U = \bigcup_{i=1}^{N} I_i$.

Then $U \triangle E \subseteq (G \setminus E) \cup \left( \bigcup_{i=N+1}^{\infty} I_i \right)$ of mea $< \varepsilon + \varepsilon = 2\varepsilon$, proving (vi).

Appendix

$(X, \mathcal{X}, \mu)$ is called

1) a measure space if $X$ is a set and $\mathcal{X}$ is a $\sigma$-algebra (of subsets of $X$) and $\mu : \mathcal{X} \to [0, +\infty]$ is a measure ($\mu(\emptyset) = 0$ and is countably additive)

2) a probability space if it is a measure space and $\mu(X) = 1$

(Subtraction & Monotone Conv. Lemma for Measures)

Th. Let $(X, \mathcal{A}, \mu)$ be a measure space

(i) Suppose $A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(A) < +\infty$. Then

$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

(because $B = (B \setminus A) \dot{\cup} A$)

(ii) Let $A_n \subseteq A_{n+1}$ $\forall n$ and $A = \bigcup_{n \in \mathbb{N}} A_n$. Suppose $A_n \in \mathcal{A}$ $\forall n$.

Then $\mu(A_n) \uparrow_n \mu(A)$, i.e.

$\mu(A_n) \leq \mu(A_{n+1})$ $\forall n$ &

$\lim_n \mu(A_n) = \mu(A)$.

(may assume $\mu(A_n) < +\infty$ $\forall n$; why?)

Then, write $A = A_1 \cup \dot{\bigcup}_{n=1}^{\infty} (A_{n+1} \setminus A_n)$ (disjoint union of measurable sets)

$$\text{so } \mu(A) = \mu(A_1) + \sum_{n=1}^{\infty} (\mu(A_{n+1}) - \mu(A_n))$$

$$= \lim_n \mu(A_n)$$

(iii) Let $B_n \supseteq B_{n+1}$ $\forall n$ & $B := \bigcap_{n \in \mathbb{N}} B_n$. Suppose each $B_n \in \mathcal{A}$ $\forall n$. Then

$$\mu(B) = \lim_n \mu(B_n)$$

provided that $\mu(B_1) < +\infty$ (or $\mu(B_N) < +\infty$ for some $N$)

proof of (iii): Let $A_n := B_1 \setminus B_n \ \forall n$. Then each $A_n \in \mathcal{A}$, $A_n \uparrow \bigcup_{n\in\mathbb{N}} A_n = B_1 \setminus \left( \bigcap_{n\in\mathbb{N}} B_n \right) = B_1 \setminus B$

and it follows from (i) & (ii) that

$$\lim_n \left[ \mu(B_1) - \mu(B_n) \right] = \lim_n \mu(A_n) = \mu(B_1 \setminus B) = \mu(B_1) - \mu(B)$$

so $\mu(B) = \lim_n \mu(B_n)$.